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DIFFERENTIABILITY PROPERTIES OF THE FAMILY OF P-PARALLEL BODIES

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We investigate the differentiability of the quermassintegrals with respect to the one-parameter family of the p-parallel bodies. As in the classical case, we obtain that the volume is always differentiable. Although there is no polynomial expression for a p-sum, the rest of quermassintegrals are differentiable on positive values of the parameter too. We prove a sharp lower bound for the derivative of the support function of the p-inner parallel bodies along with equality conditions.

1. PRELIMINARIES AND MAIN RESULTS

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., non-empty compact convex sets in the Euclidean space $\mathbb{R}^n$, endowed with the standard scalar product $\langle\cdot,\cdot\rangle$, and let $\mathcal{K}^n_0$ be the subset of $\mathcal{K}^n$ consisting of all convex bodies containing the origin 0. We also denote by $\mathcal{K}^n_n$ (respectively, $\mathcal{K}^n_{(0)}$) the subset of $\mathcal{K}^n$ having interior points (0 as an interior point). For $M \subseteq \mathbb{R}^n$, $\operatorname{conv} M$ and $\operatorname{cl} M$ will denote its convex hull and closure, and if M is measurable, we write $\operatorname{vol}(M)$ to denote its volume, i.e., n-dimensional Lebesgue measure. Let B^n be the n-dimensional unit ball and $\mathbb{S}^{n-1}$ the $(n-1)$-dimensional unit sphere of $\mathbb{R}^n$.

The Minkowski addition and its counterpart, the Minkowski difference, of non-empty sets in $\mathbb{R}^n$ are defined, respectively, as

$$A + B = \{a + b : a \in A, b \in B\}, \quad A \sim B = \{x \in \mathbb{R}^n : B + x \subseteq A\}.$$

We refer the reader to [**17**, Section 3.1] for a detailed study of the same.

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In 1962 Firey introduced the following generalization of the classical Minkowski addition (see [**4**]). For $1\leq p<\infty$ and $K,E\in\mathcal{K}^n_0$, the p-sum (or L_p-sum) of K and E is the convex body $K+_p E\in\mathcal{K}^n_0$ defined as follows:

$$h(K+_p E,u)=\Bigl(h(K,u)^p+h(E,u)^p\Bigr)^{\frac{1}{p}},$$

for all $u\in\mathbb{S}^{n-1}$, where $h(K,u)=\max\bigl\{\langle x,u\rangle : x\in K\bigr\}$ is the support function of K in the direction u. There is a homothety product corresponding to the p-sum defined by $\lambda\cdot K:=\lambda^{1/p}K$ for $\lambda\geq 0$.

In [**12**] the following counterpart of the p-sum was introduced: for $K,E\in\mathcal{K}^n_0$, $E\subseteq K$, and $1\leq p<\infty$, the p-difference of K and E is defined as

$$K\sim_p E=\Bigl\{x\in\mathbb{R}^n : \langle x,u\rangle\leq\bigl(h(K,u)^p-h(E,u)^p\bigr)^{\frac{1}{p}},\ u\in\mathbb{S}^{n-1}\Bigr\}.$$

When $p=1$, in both above cases the usual Minkowski sum and difference are obtained. From the definition it follows that for any $1\leq p<\infty$,

$$h(K\sim_p E,u)\leq\bigl(h(K,u)^p-h(E,u)^p\bigr)^{\frac{1}{p}}.$$

When dealing with the p-difference, it is useful to work with the following subfamily of convex sets (see [**12**] for further details):

$$\mathcal{K}^n_{00}(E)=\bigl\{K\in\mathcal{K}^n_0 : 0\in K\sim \mathrm{r}(K;E)E\bigr\},$$

where $\mathrm{r}(K;E)=\max\bigl\{r\geq 0 : x+rE\subseteq K \text{ for some } x\in\mathbb{R}^n\bigr\}$ is the relative inradius of K with respect to E.

Let $E\in\mathcal{K}^n_0$ and $K\in\mathcal{K}^n_{00}(E)$. The *full system of p-parallel bodies* of K relative to E, $1\leq p<\infty$, is defined as follows.

Definition 1 ([12]). Let $E\in\mathcal{K}^n_0$ and $K\in\mathcal{K}^n_{00}(E)$. For $1\leq p<\infty$,

$$K^p_\lambda=\begin{cases} K\sim_p|\lambda|E & \text{if}\quad -\mathrm{r}(K;E)\leq\lambda\leq 0,\\ K+_p\lambda E & \text{if}\quad 0\leq\lambda<\infty.\end{cases}$$

K^p_λ is the *p-inner* (respectively, *p-outer*) parallel body of K at distance $|\lambda|$ relative to E and $K^p_{-\mathrm{r}(K;E)}$ is the *p-kernel of K with respect to E*.

The p-kernel of $K\in\mathcal{K}^n_{00}(E)$ is always a degenerate convex body for all $1\leq p<\infty$ (see [**2**, p. 59] for $p=1$ and [**12**, Proposition 3.1] for $p>1$).

Differentiability properties of functions that depend on one-parameter families of convex bodies play an important role in some proofs in Convex Geometry (see e.g. [**17**, Theorem 7.6.19 and Notes to Section 7.6]). In particular, for $E\in\mathcal{K}^n_n$ and $K\in\mathcal{K}^n$, the differentiability of functions depending on the full system of 1-parallel bodies was already addressed by Bol [**1**] and Hadwiger [**6**]. In this case, i.e., when $p=1$, the considered functions are the (relative) quermassintegrals $W_i(K^1_\lambda;E)$, $i=0,\dots,n-1$ (see Section 2 for a precise description).

One of the most useful classical tools in this context is the differentiability of the function $\mathrm{vol}(K_\lambda^1)$ on $-\mathrm{r}(K;E)\leq\lambda\leq 0$, and the following consequence of its explicit computation:

$$\mathrm{vol}(K)=n\int_{-\mathrm{r}(K;E)}^{0}W_1(K_\lambda^1;E)\,\mathrm{d}\lambda. \tag{1}$$

Further results and applications of the differentiability of quermassintegrals with respect to the one-parameter family of 1-parallel bodies can be found in [**10**] and the references therein.

In this work we approach the differentiability of the (relative) quermassintegrals $W_i(K_\lambda^p;E)$ as functions of the parameter $\lambda\in\bigl(-\mathrm{r}(K;E),\infty\bigr)$. We prove that they are always differentiable on $[0,\infty)$, providing an explicit expression for the derivative, while, in general, we only have differentiability almost everywhere on $\bigl(-\mathrm{r}(K;E),0\bigr)$. For the sake of brevity we write $W_i(\lambda)=W_i(K_\lambda^p;E)$; if the distinction of p is necessary we write $W_i(\lambda;p)$.

Proposition 2. *Let $E\in\mathcal{K}_0^n$, $K\in\mathcal{K}_{00}^n(E)$ and $1\leq p<\infty$. Then $W_i(\lambda)$ is differentiable with the exception of at most countably many points on $\bigl(-\mathrm{r}(K;E),0\bigr)$, $0\leq i\leq n-1$, and*

$$\frac{\mathrm{d}^-}{\mathrm{d}\lambda}W_i(\lambda)\geq\frac{\mathrm{d}^+}{\mathrm{d}\lambda}W_i(\lambda)\geq|\lambda|^{p-1}(n-i)W_{p,i}(\lambda,E;E).$$

Here, $W_{p,i}(\lambda,E;E):=W_{p,i}(K_\lambda^p,E;E)$ (see (4)) is defined via a variational argument involving p-sums. We refer to Section 2, especially to Theorem 5, for the precise definition and references in the literature. We notice that the differentiability of $W_i(\lambda)$ does not imply, in general, that the lower bound is attained (see Remark 16).

In order to get similar properties on the range $(0,\infty)$, first it will be shown that, for $\lambda\geq 0$, and wherever both one-sided derivatives exist,

$$\frac{\mathrm{d}^-}{\mathrm{d}\lambda}W_i(\lambda)\geq\frac{\mathrm{d}^+}{\mathrm{d}\lambda}W_i(\lambda)$$

(Proposition 17). Then, proving that the above lower bound for the right derivative also holds in this case, we will get our main result.

Theorem 3. *Let $E\in\mathcal{K}_{(0)}^n$, $K\in\mathcal{K}_{00}^n(E)$ and let $1\leq p<\infty$. Then $W_i(\lambda)$ is differentiable on $(0,\infty)$, $0\leq i\leq n-1$, and*

$$W_i'(\lambda)=\lambda^{p-1}(n-i)W_{p,i}(\lambda,E;E).$$

As usual, when we write f' for a function f, we mean that the left and right derivatives exist and coincide.

As a consequence of deep known results of Lutwak [**11**] relating the volume and the p-sum of convex bodies, we establish in Theorem 24 that $\mathrm{vol}(K_\lambda^p)$ is differentiable on $\bigl(-\mathrm{r}(K;E),\infty\bigr)$, providing an explicit expression for its derivative.

In the last part of the paper we deal with the differentiability of the support function $h(\lambda,u):=h(K_\lambda^p,u)$ in terms of λ:

Theorem 4. *Let $E \in \mathcal{K}^n_{(0)}$, $K \in \mathcal{K}^n_{00}(E)$ and let $1 \le p < \infty$. Then, for all $u \in \mathbb{S}^{n-1}$,*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} h(\lambda, u) \ge \frac{|\lambda|^{p-1} h(E,u)^p}{h(\lambda,u)^{p-1}} \tag{2}$$

almost everywhere on $\bigl(-\mathrm{r}(K;E),0\bigr]$. Equality holds for all $u \in \mathbb{S}^{n-1}$, almost everywhere on $\bigl[-\mathrm{r}(K;E),0\bigr]$, if and only if $K = K^p_{-\mathrm{r}(K;E)} +_p \mathrm{r}(K;E)E$.

The paper is organized as follows. In the next section we introduce the notions and results which are used throughout the paper along with specific notation and references. In Section 3 we study the differentiability of the quermassintegrals in the above mentioned sense, proving Proposition 2 and Theorem 3, as well as the differentiability of the volume in Theorem 24. Finally, in Section 4 we prove Theorem 4 and a consequence of it.

2. GENERAL BACKGROUND

For convex bodies $K_1, \dots, K_m \in \mathcal{K}^n$ and real numbers $\lambda_1, \dots, \lambda_m \ge 0$, the volume of the linear combination $\lambda_1 K_1 + \cdots + \lambda_m K_m$ is expressed as a polynomial of degree at most n in the variables $\lambda_1, \dots, \lambda_m$,

$$\mathrm{vol}\bigl(\lambda_1 K_1 + \cdots + \lambda_m K_m\bigr) = \sum_{i_1,\dots,i_n=1}^{m} V(K_{i_1}, \dots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n},$$

whose coefficients $V(K_{i_1}, \dots, K_{i_n})$ are the *mixed volumes* of $K_1, \dots, K_m$. Notice that such a polynomial expression is not possible for the sum $+_p$ when $p > 1$ (see e.g. [**5**]). Further, it is known that there exist finite Borel measures on $\mathbb{S}^{n-1}$, the *mixed area measures* $S(K_2, \dots, K_n, \cdot)$, such that

$$V(K_1, \dots, K_n) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(K_1, u) \,\mathrm{d}S(K_2, \dots, K_n, u).$$

If only two convex bodies $K, E \in \mathcal{K}^n$ are involved in the above sum, the mixed volumes arising $V\bigl(K[n-i], E[i]\bigr) = W_i(K;E)$ are called the *quermassintegrals* of K (relative to E), and $[i]$ to the right of a convex body indicates that it appears i times. In particular, we have $W_0(K;E) = \mathrm{vol}(K)$ and $W_n(K;E) = \mathrm{vol}(E)$. We notice that

$$W_i(K;E) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(K,u) \,\mathrm{d}S\bigl(K[n-i-1], E[i], u\bigr). \tag{3}$$

If $K, E \in \mathcal{K}^n_0$, using a variational argument involving the p-sum, other functionals can be introduced. This is the case, for example, of the so-called mixed quermassintegrals defined by Lutwak in [**11**]; for further functionals defined in such a variational way, we refer to [**17**, Section 9.1]. The following theorem gathers deep results in the L_p-Brunn-Minkowski theory on which some of the proofs of this paper are based on. Note that we need the stronger assumption $K, L \in \mathcal{K}^n_{(0)}$ and $E \in \mathcal{K}^n_n$ in order the integral expression to make sense.

Theorem 5 ([17, Theorems 9.1.1 and 9.1.2], [11]). *Let* $K, L \in \mathcal{K}^n_{(0)}$ *and* $E \in \mathcal{K}^n_n$. *Let* $1 \le p < \infty$ *and* $0 \le i \le n-1$. *Then*

$$\begin{aligned} \frac{n-i}{p} W_{p,i}(K,L;E) &:= \lim_{\varepsilon\to 0^+} \frac{W_i(K +_p \varepsilon \cdot L; E) - W_i(K;E)}{\varepsilon} \\ &= \frac{n-i}{p}\frac{1}{n}\int_{\mathbb{S}^{n-1}} h(L,u)^p h(K,u)^{1-p}\,\mathrm{d}S\big(K[n-i-1], E[i], u\big). \end{aligned} \tag{4}$$

Moreover,

$$W_{p,i}(K,L;E)^{n-i} \ge W_i(K;E)^{n-i-p} W_i(L;E)^p \tag{5}$$

and

$$W_i(K +_p L; E)^{\frac{p}{n-i}} \ge W_i(K;E)^{\frac{p}{n-i}} + W_i(L;E)^{\frac{p}{n-i}}. \tag{6}$$

The following binary operation on the real numbers was introduced in [**12**] in order to deal with p-parallel bodies. Since we will often use it along this work, we detail it here for completeness. Let $+_p : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ denote the binary operation defined by

$$a +_p b = \begin{cases} \operatorname{sgn}_2(a,b)\left(|a|^p + |b|^p\right)^{\frac{1}{p}} & \text{if } ab > 0, \\ \operatorname{sgn}_2(a,b)\left(\max\big\{|a|,|b|\big\}^p - \min\big\{|a|,|b|\big\}^p\right)^{\frac{1}{p}} & \text{if } ab \le 0, \end{cases}$$

being $\operatorname{sgn}_2 : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ the function given by

$$\operatorname{sgn}_2(a,b) = \begin{cases} \operatorname{sgn}(a) = \operatorname{sgn}(b) & \text{if } ab > 0, \\ \operatorname{sgn}(a) & \text{if } ab \le 0 \text{ and } |a| \ge |b|, \\ \operatorname{sgn}(b) & \text{if } ab \le 0 \text{ and } |a| < |b|; \end{cases}$$

as usual, sgn denotes the sign function and $0 +_p 0 := 0$. For $\lambda \ge 0$ and $a \in \mathbb{R}$, we will also use the product $\lambda \cdot a := \lambda^{1/p} a$.

For $ab > 0$, this definition corresponds essentially to the classical p-mean ([**7**, Chapter II]) but does not correspond to any of the more general ϕ-means considered in [**7**, Chapter III].

Commutativity, associativity and distributivity of $+_p$ can be easily proved distinguishing the sign of the involved real numbers (see [**12**]).

Lemma 6. *Let* $a, b, c \in \mathbb{R}$. *Then*

(i) $a +_p b = b +_p a$,

(ii) $(a +_p b) +_p c = a +_p (b +_p c) = (a +_p c) +_p b$,

(iii) $a(b +_p c) = (ab) +_p (ac)$.

The following inequality between real numbers can be easily obtained as a consequence of the mean value theorem applied to the function t^p. It will be useful later.

Lemma 7. *Let* $0 \leq a \leq b$ *and* $1 \leq p < \infty$. *Then,*

$$p\,(b-a)a^{p-1} \leq b^p - a^p \leq p\,(b-a)b^{p-1}. \tag{7}$$

We will be dealing with functions concerning p-parallel bodies, which instead of being concave, satisfy an analogous inequality involving $+_p$. In order to address this property we will name it $+_p$-concavity in the following definition. We notice that given an interval $I \subseteq \mathbb{R}$, $x, y \in I$ and $\lambda \in [0,1]$, it follows from [**12**, Lemma 4.1] that $(1-\lambda)\cdot x +_p \lambda \cdot y \in I$.

Definition 8. Let $f : I \longrightarrow \mathbb{R}$, with $I \subseteq \mathbb{R}$ an interval, and let $1 \leq p < \infty$. We say that f is $+_p$*-concave* if for all $x, y \in I$ and $\lambda \in [0,1]$,

$$f\big((1-\lambda)\cdot x +_p \lambda \cdot y\big) \geq (1-\lambda)f(x) + \lambda f(y).$$

We say that f is $+_p$*-convex* if $-f$ is $+_p$-concave.

If $p = 1$ this is the usual definition of concavity. $+_p$-concave functions are not as *nice* as concave functions. However, sometimes they share their good properties. Next we prove the existence of derivatives almost everywhere (cf. [**17**, Theorem 1.5.4]), as well as absolute continuity (cf. [**14**, Remark B, p. 13]) for monotone $+_p$-concave functions in appropriate intervals, since they are indeed concave.

Lemma 9. *Let* $f : I \longrightarrow \mathbb{R}$ *be an increasing* $+_p$*-concave function,* $1 \leq p < \infty$, *with* $I \subseteq (-\infty, 0]$ *an interval. Then* f *is a concave function.*

Proof. Let $x, y \in I$ and $\lambda \in [0,1]$. Using the concavity of t^p for $t \geq 0$ we get

$$(1-\lambda)\cdot x +_p \lambda \cdot y = -\big((1-\lambda)(-x)^p + \lambda(-y)^p\big)^{\frac{1}{p}} \leq (1-\lambda)x + \lambda y,$$

and since f is increasing and $+_p$-concave, we get that f is concave on I. □

Next we prove that $+_p$-concave functions are quasi-concave (see e.g. [**17**, p. 520] for details), although there is no direct relation between $+_p$-concave functions and concave ones.

Lemma 10. *Let* $I \subseteq \mathbb{R}$ *be an interval and let* $1 \leq p < \infty$. *If* $f : I \longrightarrow \mathbb{R}$ *is* $+_p$*-concave, then* f *is quasi-concave.*

Proof. The intermediate value theorem ensures that there exists $\mu_\lambda \in [0,1]$ such that $(1-\lambda)x + \lambda y = (1-\mu_\lambda)\cdot x +_p \mu_\lambda \cdot y$. Therefore,

$$\begin{aligned} f\big((1-\lambda)x + \lambda y\big) &= f\big((1-\mu_\lambda)\cdot x +_p \mu_\lambda \cdot y\big) \\ &\geq (1-\mu_\lambda)f(x) + \mu_\lambda f(y) \geq \min\big\{f(x), f(y)\big\}. \end{aligned}$$

□

Remark 11. *In general, there is no relation between $+_p$-concavity and concavity. Indeed, let $f(x)=x^p$, $p>1$, which is a convex function on $[0,\infty)$. Then:*

(i) *f is $+_q$-convex (and not $+_q$-concave) if $1\leq q<p$.*

(ii) *f is $+_q$-concave (and not $+_q$-convex) if $p<q<\infty$.*

(iii) *f is $+_p$-linear, i.e., $f\big((1-\lambda)\cdot x+_p\lambda\cdot y\big)=(1-\lambda)f(x)+\lambda f(y)$, for all $x,y\in[0,\infty)$ and $\lambda\in[0,1]$.*

From now on we fix $E\in\mathcal{K}^n_0$ and $1\leq p<\infty$, and for $K\in\mathcal{K}^n$ we write $\mathrm{r}=\mathrm{r}(K;E)$. The following known relations between p-parallel bodies will be useful throughout the whole work.

Proposition 12 ([12, Proposition 4.2]). *Let $K\in\mathcal{K}^n_{00}(E)$ and let $\lambda,\mu\geq 0$. Then, the following relations hold:*

(i) $\big(K^p_\lambda\big)^p_\mu=K^p_{\lambda+_p\mu}$.

(ii) $\big(K^p_{-\lambda}\big)^p_\mu\subseteq K^p_{(-\lambda)+_p\mu}$ *for* $\lambda\leq\mathrm{r}$.

(iii) $\big(K^p_{-\lambda}\big)^p_{-\mu}=K^p_{(-\lambda)+_p(-\mu)}$ *for* $\lambda^p+\mu^p\leq\mathrm{r}^p$.

(iv) $\big(K^p_\lambda\big)^p_{-\mu}=K^p_{\lambda+_p(-\mu)}$ *for* $\mu\leq\mathrm{r}+_p\lambda$.

(v) $\lambda K^p_\sigma=\big(\lambda K\big)^p_{\lambda\sigma}$ *for* $-\mathrm{r}\leq\sigma<\infty$.

The following straightforward facts about p-inner parallel bodies will be used without further mention: for $K\in\mathcal{K}^n_{00}(E)$ and $-\mathrm{r}\leq\lambda<\infty$,

(i) $\mathrm{r}(K^p_\lambda;E)=\mathrm{r}+_p\lambda$,

(ii) $K^p_\lambda\in\mathcal{K}^n_{00}(E)$,

(iii) if $K=K^p_{-\mathrm{r}}+_p\mathrm{r}E$, then $K^p_\lambda=K^p_{-\mathrm{r}}+_p(\mathrm{r}+_p\lambda)E$ for all $\lambda\in[-\mathrm{r},0]$.

The full system of p-parallel bodies of a convex body K is continuous with respect to the Hausdorff metric (see [**17**, Section 1.8] for the definition) and satisfies a certain concavity property that will be needed later. We include the precise statement for completeness.

Theorem 13 ([12, Theorem 4.1, Proposition 4.3]). *Let $K\in\mathcal{K}^n_{00}(E)$. Then:*

(i) *K^p_λ is continuous in λ with respect to the Hausdorff metric on $\mathcal{K}^n$.*

(ii) *K^p_λ is $+_p$-concave on $\mathcal{K}^n$ with respect to inclusion, i.e., for $\lambda\in[0,1]$ and $\mu,\sigma\in\big[-\mathrm{r},\infty\big)$,*

$$(1-\lambda)\cdot K^p_\mu+_p\lambda\cdot K^p_\sigma\subseteq K^p_{(1-\lambda)\cdot\mu+_p\lambda\cdot\sigma}. \tag{8}$$

3. QUERMASSINTEGRALS OF K_λ^P AS FUNCTIONS OF λ

The problem of studying the differentiability of the quermassintegrals $W_i(K_\lambda^1)$ of a convex body K with respect to the parameter λ of definition of the full system of parallel bodies of K, in the 3-dimensional case and with respect to the Euclidean unit ball B^3, goes back to Bol, [**1**]. In [**6**], Hadwiger addressed a closely related question, providing some partial solutions to it. This last question was posed and studied for a general gauge body E and arbitrary dimension n in [**10**], where the original problem was solved. In this section we study differentiability properties of the functions $W_i(\lambda)$.

For the sake of brevity, given $a \in \mathbb{R}$ and $b \geq 0$, we denote by $\mu(a,b)$ the real number satisfying

$$\begin{aligned} &\text{either } \ a+b = a +_p \mu(a,b), \ \text{when } \mu(a,b) = (a+b) +_p (-a), \\ &\quad\text{or } \ a-b = a +_p \big(-\mu(a,b)\big), \ \text{when } \mu(a,b) = a +_p \big(-(a-b)\big). \end{aligned} \tag{9}$$

Of course $\mu(a,b)$ will strongly depend on the "size" of a and b and their signs.

First we prove a lower bound for the right derivative of $W_i(\lambda)$ with respect to λ, for the whole range of definition $[-\mathrm{r},\infty)$.

Proposition 14. *Let $E \in \mathcal{K}^n_{(0)}$, $K \in \mathcal{K}^n_{00}(E)$, $1 \leq p < \infty$ and $0 \leq i \leq n-1$. Then, wherever the right derivative exists,*

$$\frac{\mathrm{d}^+}{\mathrm{d}\lambda} W_i(\lambda) \geq |\lambda|^{p-1}(n-i)W_{p,i}(\lambda,E;E) \quad \text{on } \ [-\mathrm{r},\infty), \tag{10}$$

and equality holds if $\lambda \in [0,\infty)$.

For the proof of this result we need the following property.

Lemma 15. *Let $E \in \mathcal{K}^n_{(0)}$, $K \in \mathcal{K}^n_{00}(E)$, $1 \leq p < \infty$ and $0 \leq i \leq n-1$, and let $\lambda \in [-\mathrm{r},\infty)$ and $\varepsilon > 0$. If there exist suitable positive constants C and $c \geq \varepsilon$, not depending on ε, such that:*

(i) $K^p_{\lambda+\varepsilon} \supseteq K^p_\lambda +_p (\varepsilon C)^{1/p}E$, *then*

$$\frac{\mathrm{d}^+}{\mathrm{d}\lambda} W_i(\lambda) \geq C\,\frac{n-i}{p} W_{p,i}(\lambda,E;E);$$

(ii) $K^p_{\lambda+\varepsilon} \subseteq K^p_\lambda +_p (\varepsilon C)^{1/p}E$, *then*

$$\frac{\mathrm{d}^+}{\mathrm{d}\lambda} W_i(\lambda) \leq C\,\frac{n-i}{p} W_{p,i}(\lambda,E;E).$$

Proof. We prove (i), and thus we assume that $K^p_{\lambda+\varepsilon} \supseteq K^p_\lambda +_p (\varepsilon C)^{1/p}E$. Then, the monotonicity of the mixed volumes (see e.g. [**17**, Section 5.1]) yields

$$\frac{W_i(\lambda+\varepsilon) - W_i(\lambda)}{\varepsilon} \geq C\frac{W_i\left(K^p_\lambda +_p (\varepsilon C)^{1/p}E;E\right) - W_i(\lambda)}{\varepsilon C}$$

for $0 < \varepsilon \leq c$, and thus, computing the limit as $\varepsilon \to 0+$ and taking into account (4), we get

$$\begin{aligned}\frac{\mathrm{d}^+}{\mathrm{d}\lambda}W_i(\lambda) &\geq C \lim_{\eta\to 0^+} \frac{W_i\big(K_\lambda^p +_p \eta^{1/p}E; E\big) - W_i(\lambda)}{\eta}\\ &= C\frac{n-i}{p}W_{p,i}(\lambda, E; E).\end{aligned}$$

Item (ii) is analogous. □

Proof of Proposition 14. Let $\varepsilon > 0$ and $\alpha \in (0,1)$, and let $\mu(\lambda,\varepsilon)$ satisfy $\lambda + \varepsilon = \lambda +_p \mu(\lambda,\varepsilon)$ (cf. (9)).

First, we assume $\lambda \in [-\mathrm{r}, 0)$ and we observe that, since we aim to take limits as $\varepsilon \to 0$, we may suppose that $-\mathrm{r} \leq \lambda < \lambda + \varepsilon < 0$. In this case, $\mu(\lambda,\varepsilon) = \big(|\lambda|^p - |\lambda+\varepsilon|^p\big)^{1/p}$, and we are going to prove that

$$\mu(\lambda,\varepsilon) \geq (\varepsilon\, C_{p,\alpha,\lambda})^{\frac{1}{p}} \quad \text{for all } \; 0 < \varepsilon \leq c(p,\alpha,\lambda), \tag{11}$$

with $C_{p,\alpha,\lambda} = p(1-\alpha)|\lambda|^{p-1}$, and

$$c(p,\alpha,\lambda) = \begin{cases} \big[1 - (1-\alpha)^{1/(p-1)}\big]|\lambda| & \text{if } p > 1,\\ |\lambda| & \text{if } p = 1.\end{cases}$$

If $p = 1$, then $\mu(\lambda,\varepsilon) = \varepsilon > (1-\alpha)\varepsilon = \varepsilon\, C_{1,\alpha,\lambda}$ for all $\varepsilon \leq |\lambda| = c(1,\alpha,\lambda)$, which establishes (11) in this case. So, let $p > 1$ and $\varepsilon \leq c(p,\alpha,\lambda)$. Then

$$(1-\alpha)^{\frac{1}{p-1}}|\lambda| \leq |\lambda| - \varepsilon = |\lambda + \varepsilon|,$$

i.e., $(1-\alpha)|\lambda|^{p-1} \leq |\lambda+\varepsilon|^{p-1}$, and with Lemma 7 for $a = |\lambda+\varepsilon|$ and $b = |\lambda|$ we get that $\mu(\lambda,\varepsilon)^p = |\lambda|^p - |\lambda+\varepsilon|^p \geq p\,\varepsilon|\lambda+\varepsilon|^{p-1} \geq \varepsilon\, C_{p,\alpha,\lambda}$ for all $\varepsilon \leq c(p,\alpha,\lambda)$, which concludes the proof of (11).

Using Proposition 12 (ii) and (11), we immediately get

$$K_{\lambda+\varepsilon}^p = K_{\lambda+_p\mu(\lambda,\varepsilon)}^p \supseteq (K_\lambda^p)_{\mu(\lambda,\varepsilon)}^p = K_\lambda^p +_p \mu(\lambda,\varepsilon)E \supseteq K_\lambda^p +_p (\varepsilon\, C_{p,\alpha,\lambda})^{\frac{1}{p}}E.$$

Thus, Lemma 15 ensures that

$$\frac{\mathrm{d}^+}{\mathrm{d}\lambda}W_i(\lambda) \geq C_{p,\alpha,\lambda}\frac{n-i}{p}W_{p,i}(\lambda,E;E) = (1-\alpha)|\lambda|^{p-1}(n-i)W_{p,i}(\lambda,E;E)$$

for all $\alpha \in (0,1)$. It proves (10) when $\lambda < 0$.

If $\lambda = 0$, then writing $\eta = \varepsilon^p$ and using (4),

$$\begin{aligned}\frac{\mathrm{d}^+}{\mathrm{d}\lambda}\bigg|_{\lambda=0} W_i(\lambda) &= \lim_{\varepsilon\to 0^+}\varepsilon^{p-1}\lim_{\eta\to 0^+}\frac{W_i\big(0 +_p \eta^{1/p}\big) - W_i(0)}{\eta}\\ &= \begin{cases} 0 & \text{if } p > 1,\\ (n-i)W_{1,i}(0,E;E) & \text{if } p = 1.\end{cases}\end{aligned}$$

Therefore (10) holds with equality.

Next, we assume $\lambda > 0$. Now $\mu(\lambda,\varepsilon) = \big((\lambda+\varepsilon)^p - \lambda^p\big)^{1/p}$, and therefore, Lemma 7 yields

$$(12) \qquad \big(p\varepsilon\lambda^{p-1}\big)^{\frac{1}{p}} \leq \mu(\lambda,\varepsilon) \leq \big(p\varepsilon(\lambda+\varepsilon)^{p-1}\big)^{\frac{1}{p}}.$$

Using Proposition 12(i), the left inequality in (12) implies

$$\begin{aligned} K^p_{\lambda+\varepsilon} &= K^p_{\lambda+_p\mu(\lambda,\varepsilon)} = (K^p_\lambda)^p_{\mu(\lambda,\varepsilon)} \supseteq K^p_\lambda +_p \big(\varepsilon p\lambda^{p-1}\big)^{\frac{1}{p}} E \\ &\supseteq K^p_\lambda +_p \big(\varepsilon(1-\alpha)p\lambda^{p-1}\big)^{\frac{1}{p}} E \end{aligned}$$

for all $\varepsilon > 0$, and Lemma 15 yields

$$\frac{\mathrm{d}^+}{\mathrm{d}\lambda} W_i(\lambda) \geq (1-\alpha)\lambda^{p-1}(n-i)W_{p,i}(\lambda,E;E)$$

for any $\alpha \in (0,1)$. It shows (10) on $(0,\infty)$.

Next we deal with the equality case. Noticing that $(\lambda+\varepsilon)^{p-1} \leq (1+\alpha)\lambda^{p-1}$ if and only if $\varepsilon \leq \lambda\left[(1+\alpha)^{1/(p-1)} - 1\right]$, we get from the right inequality in (12) that

$$\mu(\lambda,\varepsilon) \leq \big(\varepsilon p(1+\alpha)\lambda^{p-1}\big)^{\frac{1}{p}},$$

and hence, by Proposition 12(i), that

$$(13) \qquad K^p_{\lambda+\varepsilon} = K^p_\lambda +_p \mu(\lambda,\varepsilon)E \subseteq K^p_\lambda +_p \big(\varepsilon p(1+\alpha)\lambda^{p-1}\big)^{\frac{1}{p}} E$$

for $\varepsilon \leq \lambda\left[(1+\alpha)^{1/(p-1)} - 1\right]$. Now, applying Lemma 15 we obtain

$$\frac{\mathrm{d}^+}{\mathrm{d}\lambda} W_i(\lambda) \leq (1+\alpha)\lambda^{p-1}(n-i)W_{p,i}(\lambda,E;E)$$

for any $\alpha \in (0,1)$ which, together with (10), proves the equality case and concludes the proof. □

Remark 16. *We notice that if we work on the range* $(-\mathrm{r},0)$*, the inclusion in* (13) *would be reversed, and we cannot expect to get equality in* (10).

We are now ready to prove Proposition 2.

Proof of Proposition 2. Expressions (6), (8) imply that the function $W_i(\lambda)^{p/(n-i)}$ is $+_p$-concave and increasing on $(-\mathrm{r},0)$. Then, Lemma 9 ensures that it is concave on this range. Hence there exist left and right derivatives of $W_i(\lambda)$ and they satisfy the required inequality on $(-\mathrm{r},0)$. Finally, (10) concludes the proof. □

The next result cannot be obtained as a consequence of the $+_p$-concavity of the full system of p-parallel bodies (8), since there is no analogue of Lemma 9 for $+_p$-concave increasing functions defined on $[0,\infty)$ (see Remark 11).

Proposition 17. *Let $E\in\mathcal{K}^n_0$, $K\in\mathcal{K}^n_{00}(E)$, $1\leq p<\infty$ and $0\leq i\leq n-1$. Then, wherever the left derivative exists for $\lambda\geq 0$,*

$$\frac{\mathrm{d}^-}{\mathrm{d}\lambda}W_i(\lambda)\geq\frac{\mathrm{d}^+}{\mathrm{d}\lambda}W_i(\lambda).$$

Proof. By (8) and Lemma 6, it is easy to check that

$$K^p_{\lambda+_p(-t)}+_p K^p_{\lambda+_p t}\subseteq 2^{\frac{1}{p}}K^p_\lambda \tag{14}$$

for all $t>0$ such that $\lambda+_p(-t)>-\mathrm{r}$. Then, (6) yields

$$W_i\big(2^{1/p}K^p_\lambda;E\big)^{\frac{p}{n-i}}\geq W_i\big(\lambda+_p(-t)\big)^{\frac{p}{n-i}}+W_i(\lambda+_p t)^{\frac{p}{n-i}},$$

which, by the homogeneity of W_i amounts to

$$W_i(\lambda)^{\frac{p}{n-i}}-W_i\big(\lambda+_p(-t)\big)^{\frac{p}{n-i}}\geq W_i(\lambda+_p t)^{\frac{p}{n-i}}-W_i(\lambda)^{\frac{p}{n-i}}. \tag{15}$$

Let $\varepsilon>0$ with $-\mathrm{r}<\lambda-\varepsilon$. By (9) we write $\lambda-\varepsilon=\lambda+_p\big(-\mu(\lambda,\varepsilon)\big)>-\mathrm{r}$, and with

$$m(a,b):=\frac{W_i(b)^{p/(n-i)}-W_i(a)^{p/(n-i)}}{W_i(b)-W_i(a)},$$

inequality (15) implies that

$$\begin{aligned}W_i(\lambda)-W_i(\lambda-\varepsilon)&=\frac{W_i(\lambda)^{p/(n-i)}-W_i(\lambda-\varepsilon)^{p/(n-i)}}{m(\lambda-\varepsilon,\lambda)}\\&\geq\frac{W_i\big(\lambda+_p\mu(\lambda,\varepsilon)\big)^{p/(n-i)}-W_i(\lambda)^{p/(n-i)}}{m(\lambda-\varepsilon,\lambda)}\\&=\Big(W_i\big(\lambda+_p\mu(\lambda,\varepsilon)\big)-W_i(\lambda)\Big)\frac{m\big(\lambda,\lambda+_p\mu(\lambda,\varepsilon)\big)}{m(\lambda-\varepsilon,\lambda)}.\end{aligned} \tag{16}$$

We notice that $m(a,b)$ is the slope in $\mathbb{R}^2$ of the straight line joining the points $\big(W_i(a),W_i(a)^{p/(n-i)}\big)$ and $\big(W_i(b),W_i(b)^{p/(n-i)}\big)$, which yields

$$\lim_{a\to b^-}m(a,b)=\lim_{c\to b^+}m(b,c)=\frac{p}{n-i}W_i(b)^{\frac{p}{n-i}-1}. \tag{17}$$

In order to compute the limit in (16) we need to control the size of the right-hand side in the latter inequality. Since $\mu(\lambda,\varepsilon)=\big(\lambda^p-(\lambda-\varepsilon)^p\big)^{1/p}$, given $\alpha\in(0,1)$, an easy computation proves that, for ε small enough,

$$\lambda+_p\mu(\lambda,\varepsilon)=\big(2\lambda^p-(\lambda-\varepsilon)^p\big)^{\frac{1}{p}}\geq\lambda+(1-\alpha)\,\varepsilon. \tag{18}$$

Indeed, if $\lambda=0$, then (18) is valid for all $\varepsilon>0$, whereas if $\lambda>0$ it suffices to consider

$$\varepsilon\in\left(0,\lambda\,\frac{1-(1-\alpha)^{1/(p-1)}}{1+(1-\alpha)^{p/(p-1)}}\right].$$

Thus, for $\varepsilon > 0$ small enough we get

$$\frac{W_i(\lambda) - W_i(\lambda - \varepsilon)}{\varepsilon} \geq \frac{W_i\big(\lambda + (1-\alpha)\,\varepsilon\big) - W_i(\lambda)}{\varepsilon} \, \frac{m\big(\lambda, \lambda +_p \mu(\lambda, \varepsilon)\big)}{m(\lambda - \varepsilon, \lambda)}.$$

Then, taking limits as $\varepsilon \to 0+$ to the right in the above inequality, since, by (17), $\lim_{\varepsilon \to 0^+} m\big(\lambda, \lambda +_p \mu(\lambda, \varepsilon)\big)/m(\lambda - \varepsilon, \lambda) = 1$, we obtain

$$\begin{aligned} \frac{\mathrm{d}^-}{\mathrm{d}\lambda} W_i(\lambda) &\geq (1-\alpha) \lim_{\varepsilon \to 0^+} \frac{W_i\big(\lambda + (1-\alpha)\,\varepsilon\big) - W_i(\lambda)}{(1-\alpha)\varepsilon} \\ &= (1-\alpha) \lim_{\eta \to 0^+} \frac{W_i(\lambda + \eta) - W_i(\lambda)}{\eta} = (1-\alpha) \frac{\mathrm{d}^+}{\mathrm{d}\lambda} W_i(\lambda) \end{aligned}$$

for all $\alpha \in (0, 1)$. We notice that the above expression can be written because the right derivative always exists on $[0, \infty)$ (Proposition 14). □

We observe that, for $\lambda < 0$, (14) does not hold in general.

At this point we notice that, in the classical case $p = 1$, the differentiability of $W_i(\lambda; 1)$ on $(0, \infty)$, $0 \leq i \leq n-1$, follows immediately from the fact that $W_i(K + \lambda E; E)$ can be written as a polynomial in $\lambda \geq 0$ (see e.g. [**17**, Theorem 5.1.7]).

In order to establish the differentiability of $W_i(\lambda)$ on $(0, \infty)$, and taking into account Proposition 17, we will prove that the bound for the right derivative given in (10) provides also an upper bound for the left derivative.

Proof of Theorem 3. We are going to prove that

$$\frac{\mathrm{d}^-}{\mathrm{d}\lambda} W_i(\lambda) \leq \lambda^{p-1}(n-i) W_{p,i}(\lambda, E; E), \tag{19}$$

which, together with the equality case in Proposition 14 and Proposition 17, will conclude the proof.

Let $\lambda > 0$ and $\varepsilon > 0$ with $\lambda - \varepsilon > 0$, and let $\mu(\lambda, \varepsilon) = \big(\lambda^p - (\lambda - \varepsilon)^p\big)^{1/p}$, which satisfies $\lambda - \varepsilon = \lambda +_p \big(-\mu(\lambda, \varepsilon)\big)$ (cf. (9)). From Lemma 7 we obtain that $\mu(\lambda, \varepsilon) \leq \big(p\varepsilon\lambda^{p-1}\big)^{1/p}$, and hence

$$\lambda - \varepsilon \geq \lambda +_p \left[-\big(p\varepsilon\lambda^{p-1}\big)^{\frac{1}{p}}\right],$$

which implies, by Proposition 12 (iv) and the monotonicity of the mixed volumes, that for all $0 < \varepsilon < \lambda$,

$$\frac{W_i(\lambda) - W_i(\lambda - \varepsilon)}{\varepsilon} \leq \frac{W_i(\lambda) - W_i\left(\lambda +_p \left[-\big(p\varepsilon\lambda^{p-1}\big)^{1/p}\right]\right)}{\varepsilon}. \tag{20}$$

We need some properties of the latter quermassintegral, for which we argue, where it applies, as in the proof of [**11**, Theorem (1.1)]. We show the argument for completeness. For the sake of brevity we write, for $\tau,\mu\geq 0$, $W_{1,i}(\mu,\tau):=W_{1,i}\bigl(K_\mu^p,K_\tau^p;E\bigr)$ and $\lambda(\varepsilon):=\lambda+_p\bigl[-\bigl(p\varepsilon\lambda^{p-1}\bigr)^{1/p}\bigr]$, and let

$$g(\varepsilon):=W_i\left(\lambda+_p\left[-(p\varepsilon\lambda^{p-1})^{\frac{1}{p}}\right]\right)^{\frac{1}{n-i}}=W_i\bigl(\lambda(\varepsilon)\bigr)^{\frac{1}{n-i}}.$$

We also define

$$\ell_i:=\liminf_{\varepsilon\to 0^+}\frac{W_i(\lambda)-W_{1,i}\bigl(\lambda,\lambda(\varepsilon)\bigr)}{\varepsilon},\quad \ell_s:=\limsup_{\varepsilon\to 0^+}\frac{W_{1,i}\bigl(\lambda(\varepsilon),\lambda\bigr)-W_i\bigl(\lambda(\varepsilon)\bigr)}{\varepsilon}.$$

Since $K_{\lambda(\varepsilon)}^p\subseteq K_\lambda^p$ for $\varepsilon<\lambda$, the monotonicity of the mixed volumes (cf. (4)) yields that ℓ_i and ℓ_s are the lim inf and lim sup, respectively, of nonnegative functions for $0<\varepsilon<\lambda$. Using inequality (5) we obtain

$$\begin{aligned}\ell_i&\leq\liminf_{\varepsilon\to 0^+}\frac{W_i(\lambda)-W_i(\lambda)^{(n-i-1)/(n-i)}W_i\bigl(\lambda(\varepsilon)\bigr)^{1/(n-i)}}{\varepsilon}\\&=W_i(\lambda)^{\frac{n-i-1}{n-i}}\liminf_{\varepsilon\to 0^+}\frac{W_i(\lambda)^{1/(n-i)}-W_i\bigl(\lambda(\varepsilon)\bigr)^{1/(n-i)}}{\varepsilon},\end{aligned}$$

and analogously,

$$\ell_s\geq\limsup_{\varepsilon\to 0^+}W_i\bigl(\lambda(\varepsilon)\bigr)^{\frac{n-i-1}{n-i}}\frac{W_i(\lambda)^{1/(n-i)}-W_i\bigl(\lambda(\varepsilon)\bigr)^{1/(n-i)}}{\varepsilon}.$$

The continuity of the full system of p-parallel bodies with respect to the Hausdorff metric (Theorem 13 (i)) and of the quermassintegrals W_i on $\mathcal{K}^n$ (see e.g. [**17**, p. 280]) prove that g is continuous at 0. Hence we may write

$$\begin{aligned}\ell_i&\leq W_i(\lambda)^{\frac{n-i-1}{n-i}}\liminf_{\varepsilon\to 0^+}\frac{W_i(\lambda)^{1/(n-i)}-W_i\bigl(\lambda(\varepsilon)\bigr)^{1/(n-i)}}{\varepsilon}\\&\leq W_i(\lambda)^{\frac{n-i-1}{n-i}}\limsup_{\varepsilon\to 0^+}\frac{W_i(\lambda)^{1/(n-i)}-W_i\bigl(\lambda(\varepsilon)\bigr)^{1/(n-i)}}{\varepsilon}\leq\ell_s.\end{aligned}\tag{21}$$

Moreover, using the integral expressions of W_i and $W_{1,i}$ given in (3) and (4), respectively, we can write

$$\ell_i=\liminf_{\varepsilon\to 0^+}\frac{1}{n}\int_{\mathbb{S}^{n-1}}\frac{h(\lambda,u)-h\bigl(\lambda(\varepsilon),u\bigr)}{\varepsilon}\,\mathrm{d}S\left(K_\lambda^p[n-i-1],E[i],u\right)$$

and

$$\ell_s=\limsup_{\varepsilon\to 0^+}\frac{1}{n}\int_{\mathbb{S}^{n-1}}\frac{h(\lambda,u)-h\bigl(\lambda(\varepsilon),u\bigr)}{\varepsilon}\,\mathrm{d}S\left(K_{\lambda(\varepsilon)}^p[n-i-1],E[i],u\right).$$

Since

$$\lim_{\varepsilon\to 0^+}\frac{h(\lambda,u)-h\bigl(\lambda(\varepsilon),u\bigr)}{\varepsilon}=\lambda^{p-1}h(\lambda,u)^{1-p}h(E,u)^p$$

uniformly on $\mathbb{S}^{n-1}$, the continuity of $\bigl(h(\lambda,u)-h(\lambda(\varepsilon),u)\bigr)/\varepsilon$ on $\varepsilon\in(0,\lambda)$ and the weak convergence $S\bigl(K^p_{\lambda(\varepsilon)}[n-i-1],E[i],\cdot\bigr)\to S\bigl(K^p_\lambda[n-i-1],E[i],\cdot\bigr)$ ([**17**, Theorem 4.2.1] and Theorem 13 (i)) when $\varepsilon\to 0+$ prove that

$$\ell_i=\ell_s=\frac{\lambda^{p-1}}{n}\int_{\mathbb{S}^{n-1}}h(\lambda,u)^{1-p}h(E,u)^p\,\mathrm{d}S\bigl(K^p_\lambda[n-i-1],E[i],u\bigr). \tag{22}$$

Now, since $\ell_i=\ell_s$, we get from (21) that the right derivative of g^{n-i} at 0 does exist and satisfies

$$\lim_{\varepsilon\to 0^+}\frac{g(\varepsilon)^{n-i}-g(0)^{n-i}}{\varepsilon}=(n-i)g(0)^{n-i-1}\,\frac{\mathrm{d}^+}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}g(\varepsilon).$$

It implies (cf. (21))

$$\lim_{\varepsilon\to 0^+}\frac{W_i(\lambda)-W_i\bigl(\lambda(\varepsilon)\bigr)}{\varepsilon}=(n-i)\ell_i=(n-i)\ell_s. \tag{23}$$

Thus, (20), (23), (22), and (4) yield

$$\begin{aligned}\frac{\mathrm{d}^-}{\mathrm{d}\lambda}W_i(\lambda)&=\lim_{\varepsilon\to 0^+}\frac{W_i(\lambda)-W_i(\lambda-\varepsilon)}{\varepsilon}\le\lim_{\varepsilon\to 0^+}\frac{W_i(\lambda)-W_i\bigl(\lambda(\varepsilon)\bigr)}{\varepsilon}=(n-i)\,\ell_i\\&=\frac{n-i}{n}\lambda^{p-1}\int_{\mathbb{S}^{n-1}}h(E,u)^p h(\lambda,u)^{1-p}\,\mathrm{d}S\bigl(K^p_\lambda[n-i-1],E[i],u\bigr)\\&=(n-i)\lambda^{p-1}W_{p,i}(\lambda,E;E)\end{aligned}$$

for $\lambda>0$, which proves (19) and concludes the proof. □

We point out that none of the results proved so far provides a proof of the differentiability of W_i at $\lambda=0$. In order to deal with this we will need a slightly different approach. This will be treated in Corollary 21.

There exist families of convex bodies for which the functions $W_i(\lambda)$ are differentiable on $(-\mathrm{r},0)$, $0\le i\le n-1$. This is, for instance, the case of the so-called tangential bodies, which can be defined as follows: a convex body $K\in\mathcal{K}^n$ containing $E\in\mathcal{K}^n$, is called a *tangential body* of E, if through each boundary point of K there exists a support hyperplane to K also supporting E. We notice that if K is a tangential body of E, then $\mathrm{r}(K;E)=1$. We refer to [**17**, Section 2.2 and p. 149] for further detailed information.

In [**12**, Theorem 4.2] it was proven that K is a tangential body of E if and only if K^p_λ is homothetic to K for all $\lambda\in(-\mathrm{r},0)$. This property, the homogeneity of quermassintegrals and the differentiability of $(1-|\lambda|^p)^{1/p}$ on $(-1,0)$ immediately prove the following result. We notice that E is always assumed to be in $\mathcal{K}^n_0$, and any other assumption complements this one.

Lemma 18. *Let $E\in\mathcal{K}^n_n$ and $K\in\mathcal{K}^n_0$ be a tangential body of E, and let $1\le p<\infty$. Then $W_i(\lambda)$ is differentiable on $(-1,0)$, $0\le i\le n-1$, and*

$$W_i'(\lambda)=(n-i)|\lambda|^{p-1}\bigl(1-|\lambda|^p\bigr)^{\frac{n-i}{p}-1}W_i(0).$$

Next we prove a lemma that will be used to provide an upper bound for the left derivative of $W_i(\lambda)$, involving $W_i(\lambda)$ itself. The case $p=1$ was obtained in [**16**, Lemma 4.7].

Lemma 19. *Let $E\in\mathcal{K}^n_n$, $K\in\mathcal{K}^n_{00}(E)$ and $1\le p<\infty$. For all $-\mathrm{r}\le\lambda\le 0$,*

$$\frac{\mathrm{r}+_p\lambda}{\mathrm{r}}K\subseteq K^p_\lambda. \tag{24}$$

Equality holds for some $\lambda\in(-\mathrm{r},0)$ if and only if K is homothetic to a tangential body of E.

Proof. Since $K\in\mathcal{K}^n_{00}(E)$ we have $\mathrm{r}E\subseteq K$, which yields $\mathrm{r}h(E,u)\le h(K,u)$ for all $u\in\mathbb{S}^{n-1}$. Thus, $h(K,u)^p/\mathrm{r}^p-h(E,u)^p\ge 0$ for all $u\in\mathbb{S}^{n-1}$, and so

$$\frac{\mathrm{r}^p-|\lambda|^p}{\mathrm{r}^p}h(K,u)^p+|\lambda|^p h(E,u)^p\le h(K,u)^p,\qquad\text{for all }\ u\in\mathbb{S}^{n-1}.$$

It implies, as required, that

$$h\left(\frac{\mathrm{r}+_p\lambda}{\mathrm{r}}K+_p|\lambda|E,u\right)\le h(K,u),\qquad\text{for all }\ u\in\mathbb{S}^{n-1}.$$

The equality case is provided by [**12**, Theorem 4.2], which ensures that (24) holds with equality for some $\lambda\in(-\mathrm{r},0)$ if and only if K is homothetic to a tangential body of E. □

Now we are ready to prove the mentioned upper bound for the left derivative of $W_i(\lambda)$. The case $p=1$ of this lemma was obtained in [**8**, Lemma 2.2].

Proposition 20. *Let $E\in\mathcal{K}^n_n$, $K\in\mathcal{K}^n_{00}(E)$, $1\le p<\infty$ and $0\le i\le n-1$. Then the left derivative exists on $(-\mathrm{r},0]$ and*

$$\frac{\mathrm{d}^-}{\mathrm{d}\lambda}W_i(\lambda)\le(n-i)\frac{|\lambda|^{p-1}}{\mathrm{r}^p-|\lambda|^p}W_i(\lambda). \tag{25}$$

For $0\le i\le n-2$, equality holds almost everywhere on $(-\mathrm{r},0)$ if and only if K is homothetic to a tangential body of E.

Proof. The existence of the left derivative is assured by the concavity of W_i (see e.g. [**15**]). Let $\lambda\in(-\mathrm{r},0]$ and $\varepsilon\ge0$ be such that $-\mathrm{r}<\lambda-\varepsilon\le\lambda$. Using (9) and Proposition 12 (iii) we can write

$$K^p_{\lambda-\varepsilon}=K^p_{\lambda+_p(-\mu(\lambda,\varepsilon))}=(K^p_\lambda)^p_{-\mu(\lambda,\varepsilon)}.$$

Then, Lemma 19 and the monotonicity and homogeneity of the mixed volumes yield

$$\left(\frac{\mathrm{r}+_p\lambda+_p\left(-\mu(\lambda,\varepsilon)\right)}{\mathrm{r}+_p\lambda}\right)^{n-i}W_i(\lambda)\le W_i(\lambda-\varepsilon),$$

and thus,

$$\frac{\mathrm{d}^-}{\mathrm{d}\lambda}W_i(\lambda) = \lim_{\varepsilon\to 0^+}\frac{W_i(\lambda)-W_i(\lambda-\varepsilon)}{\varepsilon} \leq \lim_{\varepsilon\to 0^+}\frac{1-\left(\frac{\mathrm{r}^p-|\lambda-\varepsilon|^p}{\mathrm{r}^p-|\lambda|^p}\right)^{(n-i)/p}}{\varepsilon}W_i(\lambda)$$
$$= (n-i)\frac{|\lambda|^{p-1}}{\mathrm{r}^p-|\lambda|^p}W_i(\lambda).$$

Next we deal with the equality case. From Proposition 2 we know that, with the exception of at most countably many points, the function $W_i(\lambda)$ is differentiable on $(-\mathrm{r},0)$. Hence, assuming equality in (25) we can write

$$W_i'(\lambda) = (n-i)\frac{|\lambda|^{p-1}}{\mathrm{r}^p-|\lambda|^p}W_i(\lambda)$$

almost everywhere on $(-\mathrm{r},0)$. Then, for $\mu\in(-\mathrm{r},0)$,

$$\int_\mu^0 \frac{W_i'(\lambda)}{W_i(\lambda)}\,\mathrm{d}\lambda = (n-i)\int_\mu^0 \frac{|\lambda|^{p-1}}{\mathrm{r}^p-|\lambda|^p}\,\mathrm{d}\lambda,$$

and thus we obtain that

$$W_i(\mu) = \left(\frac{\mathrm{r}+_p\mu}{\mathrm{r}}\right)^{n-i}W_i(0) = W_i\left(\frac{\mathrm{r}+_p\mu}{\mathrm{r}}K;E\right). \tag{26}$$

Therefore, because of the inclusion provided by Lemma 19, we can conclude that $\big((\mathrm{r}+_p\mu)/\mathrm{r}\big)K = K_\mu^p$ for $0\leq i\leq n-2$. Now, [**12**, Theorem 4.2] implies that K is homothetic to a tangential body of E.

Conversely, if K is homothetic to a tangential body of E then (see [**12**, Theorem 4.2]) $K_\lambda^p = \big((\mathrm{r}^p-|\lambda|^p)^{1/p}/\mathrm{r}\big)K$. The homogeneity of W_i allows us to explicitly compute the derivative on $(-\mathrm{r},0)$:

$$W_i'(\lambda) = (n-i)|\lambda|^{p-1}\frac{\big(\mathrm{r}^p-|\lambda|^p\big)^{\frac{n-i}{p}-1}}{\mathrm{r}^{n-i}}W_i(0) = (n-i)\frac{|\lambda|^{p-1}}{\mathrm{r}^p-|\lambda|^p}W_i(\lambda). \qquad \square$$

We observe that the equality case in (25) when $i=n-1$ cannot be deduced from (26), and we will treat it in a different way in Theorem 25.

As a direct consequence we get the following result.

Corollary 21. *Let $E\in\mathcal{K}_n^n$, $K\in\mathcal{K}_{00}^n(E)$, $1<p<\infty$ and $0\leq i\leq n-1$. Then $W_i(\lambda)$ is differentiable at 0 and $W_i'(0)=0$.*

Proof. Using Proposition 20 we conclude that the left derivative exists at $\lambda=0$ and $(\mathrm{d}^-/\mathrm{d}\lambda)\big|_{\lambda=0}W_i(\lambda)\leq 0$. Moreover, using Proposition 14, we can assure that the right derivative of $W_i(\lambda)$ at $\lambda=0$ exists. Finally, the equality case for (10) together with Proposition 17 allows us to conclude the result:

$$0 = \left.\frac{\mathrm{d}^+}{\mathrm{d}\lambda}\right|_{\lambda=0}W_i(\lambda) \leq \left.\frac{\mathrm{d}^-}{\mathrm{d}\lambda}\right|_{\lambda=0}W_i(\lambda)\leq 0. \qquad \square$$

We observe that the above result is not true in the classical case $p=1$, since the above used bounds for the left and right derivatives are neither zero nor equal, in general.

In the following lemma we provide an *equivalent* expression for the left derivative of $W_i(\lambda)$ involving the p-sum in computing the limit.

Lemma 22. *Let $E\in\mathcal{K}^n_0$, $K\in\mathcal{K}^n_{00}(E)$, $1\le p<\infty$ and $0\le i\le n-1$. Then, for all $\lambda\in(-\mathrm{r},0)$,*

$$\frac{\mathrm{d}^-}{\mathrm{d}\lambda}W_i(\lambda)=p|\lambda|^{p-1}\lim_{\varepsilon\to0^+}\frac{W_i(\lambda)-W_i\bigl(\lambda+_p(-\varepsilon^{1/p})\bigr)}{\varepsilon}.$$

Proof. Let $\varepsilon>0$ be such that $-\mathrm{r}<\lambda-\varepsilon$ and let $\mu(\lambda,\varepsilon)=\bigl(|\lambda-\varepsilon|^p-|\lambda|^p\bigr)^{1/p}$, which satisfies $\lambda-\varepsilon=\lambda+_p\bigl(-\mu(\lambda,\varepsilon)\bigr)$ (cf. (9)). From Lemma 7 we obtain that $p\varepsilon|\lambda|^{p-1}\le\mu(\lambda,\varepsilon)^p\le p\varepsilon|\lambda-\varepsilon|^{p-1}$, and hence

$$K^p_\lambda\sim_p\bigl(p\varepsilon|\lambda|^{p-1}\bigr)^{\frac{1}{p}}E\supseteq K^p_{\lambda-\varepsilon}\supseteq K^p_\lambda\sim_p\bigl(p\varepsilon|\lambda-\varepsilon|^{p-1}\bigr)^{\frac{1}{p}}E.$$

Then, using the monotonicity of the mixed volumes we can write

$$W_i\left(\lambda+_p\bigl(-p\varepsilon|\lambda|^{p-1}\bigr)^{\frac{1}{p}}\right)\ge W_i(\lambda-\varepsilon)\ge W_i\left(\lambda+_p\bigl(-p\varepsilon|\lambda-\varepsilon|^{p-1}\bigr)^{\frac{1}{p}}\right).$$

Therefore, since the left derivative exists (see Proposition 2),

$$\begin{aligned}p|\lambda|^{p-1}\lim_{\varepsilon\to0^+}&\frac{W_i(\lambda)-W_i\left(\lambda+_p\bigl(-p|\lambda|^{p-1}\varepsilon\bigr)^{1/p}\right)}{p|\lambda|^{p-1}\varepsilon}\le\frac{\mathrm{d}^-}{\mathrm{d}\lambda}W_i(\lambda)\\&\le\lim_{\varepsilon\to0^+}p|\lambda-\varepsilon|^{p-1}\frac{W_i(\lambda)-W_i\left(\lambda+_p\bigl(-p|\lambda-\varepsilon|^{p-1}\varepsilon\bigr)^{1/p}\right)}{p|\lambda-\varepsilon|^{p-1}\varepsilon},\end{aligned}$$

which proves the result. □

The case $i=0$ can be already found in the literature, directly related to the p-sums, though not in the context of p-inner parallel bodies. In [**11**], Lutwak proved the following integral expression for a p-variation of the volume functional.

Theorem 23 ([11, Lemma (3.2)]). *Let $K,E\in\mathcal{K}^n_{(0)}$ and $1\le p<\infty$. Then,*

$$\begin{aligned}\frac{n}{p}W_{p,0}(K,E;E)&=\lim_{\varepsilon\to0}\frac{\mathrm{vol}(K+_p\varepsilon\cdot E)-\mathrm{vol}(K)}{\varepsilon}\\&=\frac{1}{p}\int_{\mathbb{S}^{n-1}}h(E,u)^p h(K,u)^{1-p}\mathrm{d}S\bigl(K[n-1],u\bigr).\end{aligned}$$

We observe that the above formula is not a particular case of (4) when $i=0$, since here the limit as $\varepsilon\to0$ is two-sided. In the case of the left limit, the result was established using a variation of the support function, which turns out to be equivalent to the p-difference considered in this work. Using Lutwak's proof for an arbitrary $-\mathrm{r} \leq \lambda \leq 0$, we prove in Theorem 24 that the volume function of the system of parallel bodies, $\mathrm{vol}(\lambda) = \mathrm{vol}(K_\lambda^p)$, is differentiable on its whole range of definition $(-\mathrm{r}, \infty)$.

Theorem 24. *Let $E \in \mathcal{K}^n_{(0)}$, $K \in \mathcal{K}^n_{00}(E)$ and $1 \leq p < \infty$. Then, for all $\lambda \in (-\mathrm{r}, \infty)$,*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{vol}(\lambda) = |\lambda|^{p-1} \int_{\mathbb{S}^{n-1}} h(E,u)^p h(\lambda,u)^{1-p} \, \mathrm{d}S\big(K_\lambda^p[n-1], u\big). \tag{27}$$

Proof. Theorems 3 and 23 ensure that $\mathrm{vol}(\lambda)$ is differentiable on $[0, \infty)$, with the desired derivative. Thus, let $\lambda \in (-\mathrm{r}, 0)$. Since $K_\lambda^p \in \mathcal{K}^n_{00}(E)$, using Proposition 2, Lemma 22 for $i = 0$ and Theorem 23, we get

$$\begin{aligned} n|\lambda|^{p-1} W_{p,0}\big(\lambda, E; E\big) \leq \frac{\mathrm{d}^+}{\mathrm{d}\lambda}\mathrm{vol}(\lambda) &\leq \frac{\mathrm{d}^-}{\mathrm{d}\lambda}\mathrm{vol}(\lambda) \\ &= |\lambda|^{p-1} \int_{\mathbb{S}^{n-1}} h(E,u)^p h(\lambda,u)^{1-p} \, \mathrm{d}S\big(K_\lambda^p[n-1], u\big) \\ &= n|\lambda|^{p-1} W_{p,0}\big(\lambda, E; E\big), \end{aligned}$$

i.e., the volume function is differentiable and satisfies (27). □

Since $\dim K_{-\mathrm{r}}^p \leq n-1$ (see [**12**, Proposition 3.1]), the latter result provides the following integral formula for the volume of K in terms of functionals evaluated on its p-inner parallel bodies (cf. (1)):

$$\begin{aligned} \mathrm{vol}(K) &= n \int_{-\mathrm{r}}^0 |\lambda|^{p-1} W_{p,0}(\lambda, E; E) \, \mathrm{d}\lambda \\ &= \int_{-\mathrm{r}}^0 |\lambda|^{p-1} \left(\int_{\mathbb{S}^{n-1}} h(E,u)^p h(\lambda,u)^{1-p} \, \mathrm{d}S\big(K_\lambda^p[n-1], u\big) \right) \mathrm{d}\lambda. \end{aligned}$$

Theorem 23 for $p = 1$ is connected to the theory of Wulff shapes. We refer to [**17**, Section 7.5] and the references therein for detailed information, in particular, to Lemma 7.5.3. It provides, in the same way we have just done, the proof of the differentiability of $W_0(\lambda; 1)$.

We observe that, if $K \in \mathcal{K}^n_0$ and $0 \leq \varepsilon \leq 1$, then $\mathrm{vol}(K +_p \varepsilon K) - \mathrm{vol}(K) \leq \mathrm{vol}(K) - \mathrm{vol}(K \sim_p \varepsilon K)$ if $p > n$, just noticing that $K +_p \varepsilon K = (1 + \varepsilon^p)^{1/p} K$ and $K \sim_p \varepsilon K = (1 - \varepsilon^p)^{1/p} K$ ([**12**, Proposition 2.1]). Therefore, the differentiability of the volume in the above sense cannot be obtained as in [**13**].

4. DIFFERENTIABILITY PROPERTIES OF THE SUPPORT FUNCTION

For $K, E \in \mathcal{K}^n$, the concavity of the family of parallel bodies of K in $-\mathrm{r} \leq \lambda < \infty$ yields concavity of the support function, as a function in $\lambda \in (-\mathrm{r}, \infty)$, which implies the existence of derivatives almost everywhere. Moreover, in [**3**] it was proved that wherever the derivative exists, it satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}h(\lambda,u)\geq h(E,u), \tag{28}$$

and equality holds for all $u\in\mathbb{S}^{n-1}$, all $\lambda\in(0,\infty)$ and almost everywhere on $(-\mathrm{r},0)$, if and only if $K=K_{-\mathrm{r}}+\mathrm{r}E$.

For $p\geq 1$, Lemma 9 ensures the existence of derivatives of $h(\lambda,u)$ almost everywhere, and it makes sense to ask for an analogue of (28) when $1\leq p<\infty$. It is the content of Theorem 4. We notice that if $\lambda\geq 0$, the existence of the derivative, as well as its explicit expression, follow from the fact that $h(\lambda,u)^p=h(0,u)^p+\lambda^p h(E,u)^p$, i.e., equality holds in (2).

Proof of Theorem 4. The existence of the derivative of $h(\lambda,u)$ almost everywhere on $(-\mathrm{r},0)$ is ensured by Lemma 9. Writing $\lambda+\varepsilon=\lambda+_p\mu(\lambda,\varepsilon)$ (cf. (9)) and using Proposition 12 (ii), we have

$$\begin{aligned}h(\lambda+\varepsilon,u)-h(\lambda,u)&\geq h\big(K_\lambda^p+_p\mu(\lambda,\varepsilon)E,u\big)-h(\lambda,u)\\&=\big[h(\lambda,u)^p+\mu(\lambda,\varepsilon)^p h(E,u)^p\big]^{\frac{1}{p}}-h(\lambda,u)\\&\geq\frac{\mu(\lambda,\varepsilon)^p h(E,u)^p}{p\big[h(\lambda,u)^p+\mu(\lambda,\varepsilon)^p h(E,u)^p\big]^{(p-1)/p}},\end{aligned}$$

where the last inequality follows from the right-hand side of (7). Since

$$\lim_{\varepsilon\to 0^+}\big[h(\lambda,u)^p+\mu(\lambda,\varepsilon)^p h(E,u)^p\big]^{\frac{p-1}{p}}=h(\lambda,u)^{p-1}$$

and $\lim_{\varepsilon\to 0^+}\mu(\lambda,\varepsilon)^p/\varepsilon=p|\lambda|^{p-1}$, we may conclude that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}h(\lambda,u)=\lim_{\varepsilon\to 0^+}\frac{h(\lambda+\varepsilon,u)-h(\lambda,u)}{\varepsilon}\geq\frac{|\lambda|^{p-1}h(E,u)^p}{h(\lambda,u)^{p-1}}.$$

Now we deal with the equality case in (2). If $K=K_{-\mathrm{r}}^p+_p\mathrm{r}E$, it is not difficult to check that $h(\lambda,u)^p=h(-\mathrm{r},u)^p+(\mathrm{r}+_p\lambda)^p h(E,u)^p$ for all $u\in\mathbb{S}^{n-1}$, and a direct computation proves that, for all $\lambda\in[-\mathrm{r},0]$ and $u\in\mathbb{S}^{n-1}$,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}h(\lambda,u)=\frac{|\lambda|^{p-1}h(E,u)^p}{h(\lambda,u)^{p-1}}.$$

Conversely, we assume that, for all $u\in\mathbb{S}^{n-1}$ and almost everywhere on $[-\mathrm{r},0]$, equality holds in (2). For $u\in\mathbb{S}^{n-1}$, we consider the function

$$\psi(\lambda):=h(\lambda,u)^p-h(-\mathrm{r},u)^p-(\mathrm{r}+_p\lambda)^p h(E,u)^p.$$

Since $h(\lambda,u)^p$ is increasing and $+_p$-concave on $(-\mathrm{r},0)$, Lemma 9 and [**14**, Problem/Remark B, p.13] yield that it is absolutely continuous. Therefore ψ is absolutely continuous on $[-\mathrm{r},0]$, and since $\psi(-\mathrm{r})=0$ and $\psi'(\lambda)=0$ almost everywhere on $[-\mathrm{r},0]$, we get that $\psi\equiv 0$ for any $u\in\mathbb{S}^{n-1}$. In particular, $\psi(0)=0$ for any $u\in\mathbb{S}^{n-1}$, which yields $K=K_{-\mathrm{r}}^p+_p\mathrm{r}E$. □

Next we will slightly relax the equality conditions in Theorem 4, for which we will impose regularity on E: a convex body $E \in \mathcal{K}^n$ is said to be *regular* if the supporting hyperplane at every boundary point is unique. This property will ensure that the support $\operatorname{supp} S\bigl(E[n-1],\cdot\bigr) = \mathbb{S}^{n-1}$ (see e.g. [**17**, Theorem 4.5.3]):

(29) If E is regular, then equality holds in (2) almost everywhere on $[-\mathrm{r},0]$ and $S\bigl(E[n-1],\cdot\bigr)$-almost everywhere on $\mathbb{S}^{n-1}$ (instead of for all $u \in \mathbb{S}^{n-1}$) if and only if $K = K^p_{-\mathrm{r}} +_p \mathrm{r}E$.

We notice that, in order to prove (29), it suffices to see that if $K, L, E \in \mathcal{K}^n$, $K \subseteq L$, with E regular, such that $h(K,u) = h(L,u)$ $S\bigl(E[n-1],\cdot\bigr)$-almost everywhere on $\mathbb{S}^{n-1}$, then $K = L$. Indeed, under these assumptions, by (3) we get $W_{n-1}(K;E) = W_{n-1}(L;E)$, and hence

$$\int_{\mathbb{S}^{n-1}} \bigl[h(L,u) - h(K,u)\bigr] \mathrm{d}S\bigl(E[n-1],u\bigr) = 0.$$

Then $h(L,u) = h(K,u)$ for all $u \in \operatorname{supp} S\bigl(E[n-1],\cdot\bigr) = \mathbb{S}^{n-1}$, and so $K = L$.

We point out that this property can be not true for an arbitrary E. Indeed, let $M := \operatorname{supp} S\bigl(E[n-1],\cdot\bigr) \subsetneq \mathbb{S}^{n-1}$ and let $u_0 \in \mathbb{S}^{n-1}\backslash M$. Since $\mathbb{S}^{n-1}\backslash M$ is open on $\mathbb{S}^{n-1}$, there exists an open neighborhood $\Omega \subseteq \mathbb{S}^{n-1}\backslash M$ of u_0, and taking $L = \operatorname{conv}\bigl\{B^n, (1+\varepsilon)u_0\bigr\}$ and $\varepsilon > 0$ small enough such that $\operatorname{cl}(L\backslash B^n) \cap \mathbb{S}^{n-1} \subseteq \Omega$, we have $h(B^n,u) = h(L,u)$ for all $u \in M$, but $L \neq B^n$.

As mentioned at the beginning of Section 3, Hadwiger proposed to determine the convex bodies for which $W_i(\lambda,1)$ is differentiable, $1 \le i \le n-1$, with $W_i'(\lambda,1) = (n-i)W_{i+1}(\lambda,1)$. In [**9, 10**] the cases $i = n-1, n-2$ were solved, respectively. We conclude the paper by using the previous discussion to solve the corresponding p-problem for $i = n-1$. It will provide also the characterization of the equality case in (10) when $i = n-1$.

Theorem 25. *Let $E \in \mathcal{K}^n_0$ be regular, $K \in \mathcal{K}^n_{00}(E)$ and $1 \le p < \infty$. Then $W_{n-1}(\lambda)$ is differentiable on $(-\mathrm{r},0)$ with $W_{n-1}'(\lambda) = |\lambda|^{p-1} W_{p,n-1}(\lambda,E;E)$, if and only if $K = K^p_{-\mathrm{r}} +_p \mathrm{r}E$.*

Proof. First we assume that $W_{n-1}'(\lambda) = |\lambda|^{p-1} W_{p,n-1}(\lambda,E;E)$. Then, integrating and using (4), Fubini's Theorem and Theorem 4 we can write

$$\begin{aligned} W_{n-1}(K) - W_{n-1}(K^p_{-\mathrm{r}}) &= \frac{1}{n}\int_{-\mathrm{r}}^0 \left(\int_{\mathbb{S}^{n-1}} \frac{|\lambda|^{p-1} h(E,u)^p}{h(\lambda,u)^{p-1}} \,\mathrm{d}S\bigl(E[n-1],u\bigr)\right) \mathrm{d}\lambda \\ &\le \frac{1}{n}\int_{\mathbb{S}^{n-1}} \left(\int_{-\mathrm{r}}^0 \frac{\mathrm{d}}{\mathrm{d}\mu}\Big|_{\mu=\lambda} h(\mu,u)\,\mathrm{d}\lambda \right) \mathrm{d}S(E[n-1],u) \\ &= W_{n-1}(K) - W_{n-1}(K^p_{-\mathrm{r}}). \end{aligned}$$

Hence, we have equality all over the above expression, and thus

$$\int_{-\mathrm{r}}^0 \frac{|\lambda|^{p-1} h(E,u)^p}{h(\lambda,u)^{p-1}} \,\mathrm{d}\lambda = \int_{-\mathrm{r}}^0 \frac{\mathrm{d}}{\mathrm{d}\mu}\Big|_{\mu=\lambda} h(\mu,u)\,\mathrm{d}\lambda$$

$S\big(E[n-1],\cdot\big)$-almost everywhere on $\operatorname{supp} S(E[n-1],\cdot)=\mathbb{S}^{n-1}$, because E is regular. From (29) we get $K=K_{-\mathrm{r}}^{p}+_p \mathrm{r}E$.

Conversely, if $K=K_{-\mathrm{r}}^{p}+_p \mathrm{r}E$ then, by (3), Theorem 4 and (4),

$$\begin{aligned} W'_{n-1}(\lambda) &= \frac{1}{n}\int_{\mathbb{S}^{n-1}} \frac{\mathrm{d}}{\mathrm{d}\mu}\bigg|_{\mu=\lambda} h(\mu,u)\,\mathrm{d}S\big(E[n-1],u\big) \\ &= \frac{1}{n}\int_{\mathbb{S}^{n-1}} \frac{|\lambda|^{p-1}h(E,u)^p}{h(\lambda,u)^{p-1}}\,\mathrm{d}S\big(E[n-1],u\big) = |\lambda|^{p-1}W_{p,n-1}(\lambda,E;E) \end{aligned}$$

for all $\lambda\in(-\mathrm{r},0)$. □

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